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ABSTRACT

This paper reports the latest in a series of studies investigating children's performance in solving division story problems involving remainders. One aspect of the work involved examining the way in which "sense-making" is involved in the interpretation of the numerical solution obtained. Subjects were 195 sixth, seventh, and eighth grade students from a large urban middle school, with a student population of approximately 40% Caucasian and 60% African-American students of all ability levels, taught in mathematics classes by volunteer teachers. Each student was administered a practical problem requiring division with remainder. Three versions of the problem were used so that although the remainder was either equal to, greater than, or less than one-half, the solution required a whole number answer one greater than the quotient. Students' responses were examined with respect to four distinct aspects: (1) solution process; (2) execution of procedures; (3) numerical answer; and (4) interpretations. Responses were analyzed in relation to a hypothesized model. Results indicate that over 70% of the students applied division or another appropriate procedure, that 61% obtained a correct numerical answer, and that 45% of the subjects responded with the augmented quotient. Few students brought "real world" knowledge to bear on the problem and on making appropriate interpretations of their numerical process. Discussion of correct solution model, incorrect solution model, influence of remainder size, and task format is provided and implications for further investigation are given. (MDH)

Sense-Making and the Solution of Division Problems Involving Remainders: An Examination of Students' Solution Processes and Their Interpretations of Solutions

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Sense-Making and the Solution of Division Problems Involving Remainders: An Examination of Middle School Students' Solution Processes and Their Interpretations of Solutions[1]

In recent years, research interest in children's ability and tendency to "make sense" of the mathematics they learn has manifested itself in many different ways, such as an interest in the connections between procedural and conceptual knowledge (e.g., Hiebert, 1986), in the meanings children impose on the mathematical symbols and procedures they learn (e.g., Resnick, 1988), and in the ways in which children connect or fail to connect school learning to everyday experience (e.g., Saxe, 1991). Interest in children's meaningful interpretations of school mathematics, however, is not a new phenomenon; it figured prominently in the writings of Dewey (1910, 1933) and Brownell (1935, 1947), among others earlier in this century.

This paper reports the latest in a series of studies investigating children's performance in solving division story problems involving remainders. One aspect of this work has involved examining the way in which "sense-making" is involved in the interpretation of the numerical solution obtained.

In addition to their utility in studying students' application of division computations to solve story problems, division story problems involving remainders are important and interesting contexts in which to study mathematical sense-making both because they are often found to be difficult and challenging for students -- as shown by state and national assessments of mathematical knowledge -- and because they are complex cognitive tasks

which have certain important features that emphasize the importance of semantic processing in their successful solution (Silver, 1988).

One source of difficulty for children, when solving division story problems involving remainders, is that the same symbolic division expression can represent different problem situations that have different answers, the determintion of which depends on aspects of the situational context and the quantities involved in the problem (Silver, 1986). Unlike the case for most other story problems encountered by students in elementary school, sense-making is not an optional activity in solving these problems because correct computation alone cannot ensure a successful solution.

The widespread failure of American students to succeed in solving problems involving whole number division with remainders has been documented through the National Assessment of Educational Progress and several state assessments. Only 24% of a national sample of 13-year-olds was able to solve correctly the following problem which appeared on the Mathematics portion of the Third National Assessment of Educational Progress (NAEP, 1983): "An army bus holds 36 soldiers. If 1,128 soldiers are being bused to their training site, how many buses are needed?" A similar division problem appeared on the 1983 version of the California Assessment Program (CAP) Mathematics Test for Grade 6 and was answered correctly by only about 35% of the sixth-graders in California In both assessments, students commonly erred by giving non-whole-number answers. In the NAEP report, the authors inferred from students' responses that over 70% of the students recognized the problem as a division problem and could have solved the problem correctly if they had related the numerical solution back to the problem task.

To better understand the basis for the observed difficulty that students have in solving division problems involving remainders, several investigations were conducted with students in grades 6, 7, and 8 (Silver 1986, 1988; Silver, Mukhopadhyay, & Gabriele, 1989). The findings of these investigations have suggested that students' failure to solve division problems with remainders can be attributed, at least in part, to their failure to relate computational results to the situation described in the problem.

In an early study (Silver, 1986), it was hypothesized that students were failing to attend to relevant information implicitly represented in the problem situation but not explicitly stated in the story text (e. g., no one is to be left behind; on some bus there may be some empty seats). Several problem variants were created that made relevant structural information more salient, and students' performance on these variants was examined. The results of that study demonstrated students' performance could be significantly enhanced by making explicit certain implicit information in the problem or the required solution. Unlike most considerations of relevant information for problem solving in elementary mathematics, the focus of attention in the study was not so much on information that would enhance the mapping between the story text and the mathematical model but rather on enhancing the mappings between and among these two reference spaces and the story situation.

Subsequent research (Silver 1988; Silver, Mukhopadhyay, & Gabriele, 1989) examined students' performance on division problem types including augmented-quotient problems, remainder-only problems and quotient-only problems. These studies examined the effects on students' performance of their solving several division problems that required the same computation and similar referential mappings. The results indicated that students' performance on each type of problem was enhanced by having students also

solve related division problems. In general, the results were consistent with the explanation that enhanced performance was due to students' increased sensitivity and attention to the relevant semantic and referential mappings involved in the target problem solution. In particular, experience with the related problems may have drawn attention to the need for mapping into either the story text representation or the story situation representation after obtaining a solution to the augmented-quotient problem through use of a mathematical model.

Taken together, these results and the assessment findings suggested that students' failure to solve the division story problems was due, at least in part, to an incomplete mapping among the relevant referential systems. In particular, it appeared that students might map successfully from the problem text to a mathematical model (in this case, a division computation to be performed), compute an answer within the domain of the mathematics model, but fail to return to the problem story text or to the story situation referent in order to determine the best answer to the question. Figure 1 presents a schematic representation of a hypothesized version of a student's unsuccessful solution attempt.

Insert Figure 1 about here

Following from this hypothesized model, it is possible to suggest that a student's successful solution attempt would be represented as follows: the solver would map from the story (natural language) text representation of the problems into a mathematical model representation, then perform the required computation within the referential system of mathematics, expressing the resulting answer with an appropriate mathematical

representation. The solver would then map the computational result back either to the story text representation or to the implied story situation (in the "real world") representation in order to decide how to treat the quotient and remainder. Through such a process, the successful solver would finally obtain suitable mathematical and natural language representations of the solution that have accompanying interpretations and validity within the referential systems of real world situations and the knowledge domain of mathematics. Figure 2 provides a schematic representation of the mappings involved in this idealized problem solution.

Insert Figure 2 about here

Until recently, evidence for these hypothesized solution models was available only indirectly from an examination of students' performances on multiple-choice test items. Direct confirmation of these models of successful and unsuccessful solutions was obtained in an interview study (Smith & Silver, 1991) in which the problem-solving and interpretation performance of eight middle school students was examined in an interview setting in which the students solved an augmented-quotient division-with-remainders problem.

The interview protocols reported by Smith & Silver (1991) also revealed some interesting facets of students' sense-making with respect to the division problem task. In addition to the finding that the students who correctly solved the problem included a sense-making step in their solution, the interviews revealed that some students, who would have answered incorrectly if the tasks were presented in a multiple-choice format, were able to offer interesting interpretations of their numerical answers. For example, one student spoke of "squishing in" the extra students, and others suggested ordering mini-vans

rather than a full bus for the extra students. This kind of situation-based thinking and reasoning remained invisible both in the multiple-choice response format used in the prior research and the kind of summary information available from the free-response solutions on the NAEP (1983) task.

The major purpose of the study presented here was to investigate, on a larger scale than the Smith and Silver (1991) research, the solution processes and interpretations of students solving an augmented quotient division-with-remainders problem, with an emphasis on analyzing responses that would support or refute hypothesized referential models (Silver et al. 1989). A free-response pencil and paper task was used in order to provide a format that could provide easily analyzable responses from a relatively large number of students. Also, this format, unlike a multiple-choice task, could provide direct access to students' thinking and reasoning about the problem. Finally, given the evidence of situation-based reasoning found in the Smith and Silver study, it was hypothesized that the number of "left overs" might influence students' situation-based interpretations and final solutions, and remainder size was chosen as a variable of interest.

Method

Sample

The sample consisted of 195 sixth, seventh, and eighth grade students from a large urban middle school, with a student population of approximately 40% Caucasian and 60% African-American students of all ability levels. The students were members of mathematics classes taught by teachers who volunteered their classes.

Task

The following problem-solving task was administered to each student in the sample:

> The Clearview Little League is going to a Pirate game. There are 540 (532 or 554) people, including players, coaches and parents. They will travel by bus and each bus holds 40 people. How many buses will they need to get to the game?

Three versions of this problem were used so that student responses would be obtained for division problems with remainder sizes equal to one-half (540 people), less than one-half (532 people) and greater than one-half (554 people).

Administration and Procedures

The task was administered to three classes at each grade level as a 15-20 minute activity during a regular class session. At each grade level, each version of the problem was administered to one class. The instructions accompanying the task directed students to show their work, to place their answer in an answer space provided, and to explain their answer in writing.

Results

Since neither grade level nor ethnic differences were of particular interest in this study, results are reported for the aggregated sample. Students' responses were examined with respect to four distinct aspects: (a) solution process, (b) execution of procedures, (c) numerical answer, and (d)

interpretations. Each aspect was examined independently, and then an additional analysis was conducted in which various combinations of these aspects were examined.

Solution Processes.

The solution process was defined as the set of procedures used by the student to obtain a numerical solution. Although there were many solution processes that could have been used to solve the problem, such as drawing pictures or forming sets, all of the students' written responses indicated use of an algorithm. The majority of students used the long division algorithm (70%), but a significant minority (20%) of students used other algorithmic procedures, such as repeated addition. Table 1 exhibits the most frequently used algorithmic procedures and the percentage of students using each procedure. An algorithm was considered appropriate if it could potentially lead to a correct solution without requiring additional procedures not in evidence in a student's response.

Insert Table 1 about here

Execution of Procedures

The execution of procedures referred to the actions taken by the student in carrying out the solution process. Since all students in the sample used an algorithmic procedure, examination of the execution of procedures was reduced to examining the correctness of the steps in the algorithm. A student's execution of procedures was judged to be correct if and only if each arithmetic operation in the procedure was executed without error.

The work of those students who used an inappropriate algorithm to solve the problem was eliminated from examination. Table 2 shows the percentage of students who were successful or unsuccessful in executing the steps of the procedures for each of the appropriate algorithms used by the students. Overall, about 61% of the students were able to perform their calculations flawlessly. The long division algorithm, though the procedure most often used by the students, was the most difficult for them to perform completely.

Insert Table 2 about here

Numerical Answer

The number written by a student in the space provided for the answer to the initial problem, "How many buses are needed?", was considered to be the student's numerical answer. Table 3 indicates the numerical answers given by the students and the percentage of the sample giving each answer.

Insert Table 3 about here

Most of the response categories are self-explanatory, inlcuding those involving the mathematical expression of the remainder as a whole number, fraction or decimal. These categories accounted for about 10% of the total number of student responses.

A few response categories deserve special attention. For example, the two responses in the category "13 and other remainder representation" appeared to involve a combination of mathematical and situation-based

knowledge -- "13 and 1 cab" and "13 and a mini-van." In the category "Other Answers," nearly one-third of the responses involved a numerical answer greater than 100 -- responses which resulted from students' incorrect execution of the decimal division algorithm.

Interpretations

Interpretations were the explanations of solutions given by students in the space provided for that purpose.

Coding. Interpretations were coded as "appropriate", "inappropriate" or "no interpretation". An interpretation was coded as appropriate if, in the written explanation, the student said that a whole number of buses was needed because a fraction of a bus did not make sense or that there were some people who would not be able to go if an extra bus was not provided. In addition, if a student suggested that the fractional remainder represented a mini-van or gave some other reasonable meaning to the numerical answer, the explanation was coded as an appropriate interpretation. An interpretation was coded as inappropriate if a student explained the numerical answer by applying rounding or estimating rules, offered an incomplete or incorrect explanation, or otherwise gave evidence of confusion. A response was coded as "no response" if it was simply an explanation of the procedures used to find the solution, if it was a statement commenting generally or vaguely on the problem or the answer, or if no written explanation was provided.

To ensure inter-rater reliability a sample of approximately one-third of the responses was coded independently by each of two raters. An acceptably high degree of agreement between raters was obtained (Kappa = .94).

Appropriate Interpretations. About one-third of the students gave responses that were classified as appropriate interpretations. Examples of interpretations considered appropriate include one student who wrote, "You'll need 13 and a third buses. Since buses don't come in thirds, you get a whole other bus," and another student who wrote, "14 to hold everyone, and you would have empty seats for more people who decided to come."

Some appropriate explanations were provided for final answers other than 14. For example, one student who gave a final answer of 13 1/2, wrote: "520 people are riding a big bus, and you'd have to get a van for the other 20 [people]." Some students who provided final answers of 13 1/2 gave interpretations such as, "you need 13 buses and 1 van [cab or minibus]."

Inappropriate Interpretations. Only 9% of the sample gave explanations that were classified as inappropriate. Examples of inappropriate explanations include one student who wrote: "[The answer is] 14 buses because there's left over people and if you add a zero you will get 130 buses so you sort of had to estimate. Are we allowed to add zeros?" Another student, after attempting decimal division, reported: "[The answer is] 14. I got 13,065 but just looking at the number I wouldn't get that so I took the first 2 digits and added 1 because about 5 [students] would be left."

No Interpretation. More than half of the responses for the entire sample were classified in this category. In addition to papers containing no written explanation, this category also included those papers containing only general comments on the problem or the answer, such as the student who obtained an answer of 133 and wrote, "I was agassed [sic] at how many times it [the divisor] went in."

Procedural explanations -- such as, "I divided 40 into 540 and there is a remainder of 20, then I reduced 20/40 to 10/20, and then I took off the zero

and came up with 13 1/2" – were quite evident in the set of responses in this category. In fact, more than one-half (54%) of the sixth grade responses were classified as procedural explanations. In general, procedural explanations suggested students' attention to issues of mathematical form (e.g. execution of the steps of the algorithm, representational form of the numerical answer). Forty-one percent of the seventh and eighth grade students also gave detailed descriptions of their mathematical procedures, but many of these older students also provided an interpretation for their solution and their interpretations were often classified in the other categories. None of the sixth grade students, who provided a procedural description, also provided an interpretation of their numerical answer.

Response Patterns

Thus far, each aspect of a student's response has been treated independently. Determination of evidence that supports or refutes the hypothesized referential mapping models discussed earlier requires examining the interplay among the various response components. Approximately 78% of the responses provided direct evidence to support the hypothesized models, about 14% of the responses appeared to provide counter-evidence, and the remaining 8% were judged to be neutral or impossible to classify with respect to the models .

Direct supporting evidence for the hypothesized model of a correct solution was provided by approximately 32% of the responses. In particular, about 26% of the students gave a numerical answer of 14 and provided an appropriate interpretation, nearly 3% provided an answer other than 14 but also gave an appropriate interpretation, and about 3% had

flaws in the execution of their solution procedure but were able to obtain an answer of 14 and give an appropriate interpretation.

Evidence directly supporting the hypothesized model of an incorrect solution was provided by approximately 46% of the responses. In particular, about 22% of the students correctly executed an appropriate solution procedure but provided no interpretation for their incorrect numerical answer; and nearly 24% of the students incorrectly executed an appropriate procedure, gave a numerical answer other than 14 and provided no interpretation.

Counter-evidence for the hypothesized solution model was supplied by about 8% of students who were able to provide the correct numerical answer of 14 without providing any accompanying interpretation; approximately 2% who gave an inappropriate interpretation for the answer of 14; and about 4% who gave an inappropriate interpretation for an answer other than 14. Further examination of many of the cases providing the apparent counter-evidence revealed some interesting tendencies in these responses. For example, a few of the students who obtained the numerical answer 14 but provided no interpretation used a repeated addition procedure -- an algorithm which is generally not associated with direct instruction in division story problems. In the case of the students who gave an inappropriate interpretation for the answer 14, they gave explanations involving rounding and estimation -- topics also taught in the school mathematics curriculum.

Remainder Size

The written responses of the students offered no direct evidence of students being influenced by the size of the remainder in interpreting their

solutions or arriving at their final numerical answer. There was, however, evidence that remainder size interacted somewhat with success in executing procedures and in students' tendency to interpret numerical answers. Students who had the problem version with remainder size equal to one-half more often executed their computational procedures correctly than did their counterparts who had other versions of the problem. Moreover, students' responses for the problem form in which the remainder was one-half were also somewhat more likely to reveal some attempt to interpret the numerical answer.

Discussion

The major goal of this study was the examination of the solution processes and interpretations provided by students when solving an augmented quotient division-with-remainders problem, with particular emphasis on analyzing students' responses for evidence that supported or refuted the hypothesized referential mapping models proposed in earlier research (Silver et al., 1989). A secondary goal was to determine the extent to which remainder size influences students' solutions or interpretations. In this section, specific comments are made concerning each of the goals, after which some additional comments and observations are made about the findings.

Relation to Hypothesized Models

In general, the responses provided by students in this study supported the hypothesized models of correct and incorrect solutions. In particular, students' responses contain considerable evidence that the computational

requirements were not the major barrier in obtaining a correct solution, but rather that unsuccessful solutions were more often due to students' failure to engage in interpreting their computational results. In fact, over 70% of the students used the long division algorithm to solve the problem (a finding consistent with the earlier NAEP report) and an additional 20% used other appropriate procedures, and 61% of these students were able to execute their computation correctly, yet only about 45% of the subjects responded with the augmented quotient (14) as their numerical answer or were able to give an appropriate interpretation for some answer other than 14. It is worth noting that of the 45% of the students who responded with an appropriate solution, 78% also gave an appropriate interpretation for their answer.

Correct Solution Model. The disparity between the number of students who were able to compute correctly (61%) and the number who gave a correct answer (45%) relates to the hypothesized model of a correct solution discussed earlier. The findings in this study provide direct evidence to support that model. In particular, nearly 70% of the students who gave the augmented quotient (14) as their numerical answer also provided appropriate interpretations for their answer, and even a few students who gave answers other than 14 were able to give appropriate interpretations. These latter students, though small in number, were particularly interesting because they provided evidence of bringing "real world" knowledge to bear on the problem and making appropriate interpretations of their numerical responses. This group included both students who explained their numerical answers involving fractional remainders by having the fraction represent a mini-bus or van and students who explained their numerical answers involving whole number remainders by

having the remainder represent people to be arranged so as to fit in the available space. These explanations reflect students attempts to relate their outside-of-school knowledge and experiences to the numerical answers obtained from their mathematical calculations. These responses also illustrate one limitation of fixed-response formats when assessing students' problem-solving competence, at least with respect to augmented-quotient division-with-remainder problems, and the general danger in asserting that only one numerical answer to such problems can be considered correct.

The major counter-evidence for the proposed model of a correct solution was provided by the responses of those students who solved the problem using either repeated addition or subtraction or repeated multiples (which reflects a "guess and check strategy"). These students executed an appropriate mathematical procedure correctly, yet they tended not to map back to the story situation in order to provide an interpretation. It is possible, however, that these responses do not actually refute the general contention of the hypothesized models, although the responses may require some modification of a few details in the model. It seems reasonable to speculate that these procedures (repeated addition or repeated multiples), although more mathematically primitive than the long division algorithm, may be more intuitively linked to the situation described in the problem (filling up buses). Consider, for example, that adding up (or subtracting down) more naturally parallels the act of loading individuals on to a bus. Students who used these algorithms, unlike those who used long division, may have utilized these procedures as a natural consequence of their situation-based reasoning about the problem and may not have felt need to provide an explanation of their solutions. These more situationally-based

procedures, unlike the long division algorithm, implicitly contain an interpretative framework.

The only other counter-evidence to the proposed model of a correct solution was found in the responses of students who either rounded the numerical answer to get a whole number of buses or estimated the number of buses needed. These students may be attempting to make connections with other mathematical knowledge in order to make sense of their calculations. Although it could be argued that these procedures and interpretations appear to have arisen completely within the mathematical space of the problem, and that they are probably not indicative of the students mapping back to the story text or story situation to make sense of the numerical answer, one could also argue that both procedures clearly reflect the situation-based constraint that the only allowable number of buses would be a whole number.

Incorrect Solution Model. Students' responses also provided considerable evidence to support the hypothesized model of an incorrect solution. More than 20% of the students, although able to execute correctly an appropriate computational procedure, provided an incorrect numerical answer and offered no interpretation of their solution. Moreover, nearly one-quarter of the sample made computational errors that might have been corrected if the students had interpreted their final answers.

Further evidence in support of this model is obtained from the responses of nearly 10% of the students who solved the problem using long division and made a computational error involving the placement of a decimal point. This error resulted in students obtaining a quotient ten times as large as the actual numerical answer -- an error that could likely have been detected if the students had interpreted their answer. Presumably, some

students who used an inappropriate computational procedure in their attempted solution might also have detected the error if they had interpreted their solutions.

Influence of Remainder Size

With respect to the secondary goal of this study, the results provided no indication that remainder size generally appeared to influence the solution processes or interpretations provided by the middle school students in this sample. The only effect was noted for the problem form in which the remainder was one-half. Students' responses for this form were more likely than responses for the other forms to reveal successful execution of computational procedures and some attempt to interpret the numerical answer.

Further research on this issue may be needed, however, since the three problem forms were not equivalent with respect to computational complexity. In order to have a problem form with a remainder of one-half and a divisor of 40, the dividend had necessarily to be a multiple of ten. This unintentional constraint resulted in the problem being an easier computational task than the other forms. Since the students who received this form outperformed the other groups in the correct execution of computational procedures, the computational simplicity may have contributed to the finding that these students also more frequently engaged in interpreting their numerical answers than those students who solved the other forms of the problem. The apparent interaction between computational complexity and situation-based reasoning in this study is reminiscent of one of the major findings reported by Baranes, Perry and Stigler (1990).

Additional Observations

Form Versus Function. Examination of students' responses indicated that many students showed greater concern about the form in which their computations should be executed and in which their final numerical answer should be written rather than the relationship between the numerical answer and the problem being solved. For example, many students appeared to be concerned about the form in which the remainder was expressed, such as the sixth-grade student who gave 13 34/40 as her final answer and wrote as her explanation: "I got the answer and I put in [sic] a fraction because that's how our teacher taught us. Should I put the remainder in a fraction?"

Students' concern with form was also evidenced in many students' detailed, step-by-step narrative descriptions of the procedures they used to obtain their numerical answers. Moreover, attention to form further manifested itself in some students' comments regarding what they believed to be the correct way to solve the problem. Some students expressed the view that using "the correct" algorithm was the most important aspect of solving the problem. For example, one seventh grader gave a final numerical answer of 13 and wrote, "I think dividing is the correct way to answer this problem."

An excessive emphasis on particular calculation procedures or notational form is likely to impede students from correctly solving an augmented quotient problem, especially since an interpretation of the numerical response is generally needed. As noted above, except in the case of the use of alternative algorithms, students generally need to exit from the mathematical space in order to return to the story situation and

interpret their numerical answer. To engage in such processing, however, a student must perceive the need to do so. If issues of mathematical formalism are paramount in the students' attention during problem solving, then a strong motivation for interpreting the numerical result is less likely to exist. It is encouraging to note that the older students in the sample, while placing considerable emphasis on matters of mathematical form like their younger counterparts, also tended to give consideration to the interpretation of their numerical results with respect to the problem situation.

The large number of student responses emphasizing the form of procedures and answers rather than their function in solving a problem may reflect imbalances in the current emphases of a typical middle school mathematics instruction. Although it might be argued that middle school students are being taught, either explicitly or implicitly, that accurate computations and correct notational form are the most highly valued aspects of mathematics, it is equally likely that the students' responses simply reflect more localized instructional influences. Current middle school mathematics instruction is certainly dominated by attention to computational procedures, and this emphasis was reflected in the form-oriented responses of the students in this sample. Although it is encouraging that many students were able to provide interpretations and explanations that went beyond considerations of form, it is discouraging that so many other students were unable to do so. The results of this study suggest that, unless solution explanations and interpretations become a regular item on the menu of instructional activities in mathematics classrooms, it is unlikely that many students will spontaneously engage in such activity when it is appropriate to do so.

Task Format

Several students voiced objections to having to explain their answers. The objections came in one of two forms: (a) students who indicated that they had never been taught how to explain their work and that it was a difficult thing for them to do, and (b) students who indicated a belief that correct computations always produce correct answer, thereby obviating the need for further explanation. These objections -- such as the fairly typical one given by the student who wrote, "I don't know how to explain anything because there's nothing to explain. It is very hard to do this because our math teacher didn't teach us this." -- indicate the lack of experience that many middle school students have in providing written explanations for their mathematical work. Until written explanations become a more prevalent feature of mathematics assignments, students -- especially those whose writing skills may not be strong -- are likely to continue to express this lack of comfort with such tasks. Assignments that require students to provide written, or even oral, explanations for their work are consistent with calls for greater emphasis on communication in the mathematics classroom (NCTM, 1989; Silver, Kilpatrick, & Schlesinger, 1990).

The paper-and-pencil, free-response format was intended to capture students' solution processes and their situation-based thinking and reasoning. Although it appeared to function fairly well as a medium for students to communicate their thinking, there are some indications that aspects of the format and administration conditions may have had unintended consequences related to both the solution processes and the interpretations provided.

Students responses indicated the dominance of the long division algorithm as the solution procedure of choice. However, there is some anecdotal evidence to suggest that alternative algorithms may have been more prevalent than the written work revealed. For example, in post hoc interviews, some of the teachers who administered the task mentioned that some students did preliminary work on the problem on their desk tops or book covers. According to the teachers, many of the students who did this type of "scratch" work used alternative algorithms which were not transferred onto their papers. Instead, these students turned in to their teachers a final product that showed use of the long division algorithm. Moreover, from those teachers who engaged their classes in a follow-up discussion of the problem, it was learned that students not only employed a range of algorithms to solve the problem, but also used grouping and counting techniques and drew pictures. These more mathematically primitive techniques, however, were apparently judged to be unsuitable for display in their written responses.

The task format and administration conditions also appeared to influence the tendency of students to provide explanations and interpretations. Anecdotal evidence from our discussions with the teachers who participated in the study indicated that, during the discussions that followed this problem-solving activity in some classes, many students argued vigorously for alternative solutions using a variety of interpretations for the remainder and explanations of how to represent their interpretation numerically. Some students apparently argued that an extra bus was not needed because some students would be absent and would not attend the game. Others were reported to say that some kids could walk to the baseball game because the school is close to the stadium (since the school our

students attended is located less than one mile from the baseball stadium mentioned in the problem). These rich, situation-based comments about possible problem solutions were not found in the children's written responses.

The task was administered by the mathematics teacher as part of a regular mathematics class, hence students probably viewed it as a formal classroom exercise and therefore responded in a manner which they believed to be both "mathematically correct" and acceptable to their teacher. The formality of writing a response, rather than giving it orally in a class discussion, may also have contributed to the tendency of students not to reveal all of their informal thinking and reasoning about the problem. Regardless of the explanation, it seems clear from these anecdotes that the task format and administration conditions probably limited the range of students' responses. The written, free-response task format was clearly useful in revealing much about students' solution processes and interpretations, but these anecdotes suggest that the written responses may only reflect a portion of the students' thinking about the problem.

Interesting confirmation for this view comes from preliminary findings in recent work by Curcio and DeFranco (F. Curcio, written communication, January 26,1991). In their recent study, 20 middle school students solved two similar versions of the "bus" problem. The first task was presented as one of 21 interview items in which subjects were asked to review the work of another student who had solved a division-with-remainders task similar to the one in this study. The student's long division calculation and an answer in which the remainder was expressed as a fraction was presented, and the subjects were asked to comment on the result and to determine what answer they would give for the problem. On the second task, subjects

were presented with a set of facts involving numbers of persons going on a trip and the capacity of commercial vehicles available for transportation. The students were asked to make a telephone call (on a teletrainer) to order transportation for a school trip. Students had far greater success in solving the second task (17 out of 20 correct) than the first task (12 out of 20 correct). Like the students in the study reported in this manuscript, the students in the Curcio and DeFranco investigation also applied situation-based reasoning and interpretations. For example, six students ordered a number of vans and then asked for "like a car or something" to take the remaining students. Another student argued that some students would surely be absent, so it would be unnecessary to order an extra van. Although the findings reported by Curcio and DeFranco are intriguing, design limitations in their study prevent a clear determination of whether student responses are due to the effects of problem context or problem order. Nevertheless, the findings are generally compatible with the observations of the student problem-solving processes and sense-making presented in this study

From the above discussion, it appears that the task format and administration conditions may have constrained students to overemphasize the exhibition of behaviors they believe to be mathematically acceptable -- the application of formal algorithms -- and underemphasize the exhibition of behaviors they believe to be mathematically unacceptable -- situation-based reasoning and interpretations. Surely, these issues need to be considered in future assessments of students' problem solving and sense making. The findings of this study suggest that the use of alternative (i.e., non-multiple-choice) paper-and-pencil measures of students' competencies may be necessary (since gathering interview protocols is impractical) in

order to provide information needed to assess programs attempting to improve students' mathematical communication and reasoning, yet the use of alternative assessment tasks is clearly not sufficient. As the findings demonstrate, until students are more accustomed to explaining their mathematical thinking and reasoning in writing, researchers and teachers will be stymied in their efforts to gather a rich descriptive data base. More instructional attention both to sense-making and to written communication as a part of school mathematics instruction, as well as alternative assessment tasks that focus more clearly on explanation and interpretation are necessary.

Further Investigation

The findings of this study contribute to our understanding of student performance in solving division-with-remainder problems and contribute more generally to our understanding of the relationship between situation-based sense-making and mathematical problem solving. Given the findings regarding the possible limiting effect on student performance of the task setting and format used in thus study, a potentially fruitful area for further investigation involves the examination of alternative assessment settings or formats which might enhance the likelihood that students would engage in sense-making. For example, the written task used by Curcio and DeFranco , in which students were asked to examine the work of another student and determine or critique an answer, suggests possible formats that might be used to stimulate students' situation-based reasoning, since the computational burdens might be entirely removed. Of course, the findings of this study, and the earlier study by Smith and Silver (1991), also suggest that the use of interview formats may be advisable in future research.

The findings regarding students' use of alternative algorithms and the likely connection between these alternative procedures and situation-based reasoning about the problem suggests another interesting area for future investigation. In their study of problem solving across different contexts, Baranes, et al. (1990) also reported an apparent relationship between students' use of solution strategies and certain contextual features of the problem situation. Although limited in scope, the findings of Baranes, et al., taken together with the results of the study reported here, suggest the potential value of examining more closely the relationship between problem-solving procedures and situational contexts.

In addition to continued research related to the general issue of understanding how and when students connect mathematics to situations, the findings of this study suggest the wisdom of developing and implementing instructional activities in which children are challenged to engage in sense-making. Clearly, this study has shown that American students in the middle grades need more experience in explaining their mathematical solutions.

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Figure 1

Schematic representation of hypothesized unsuccessful solution

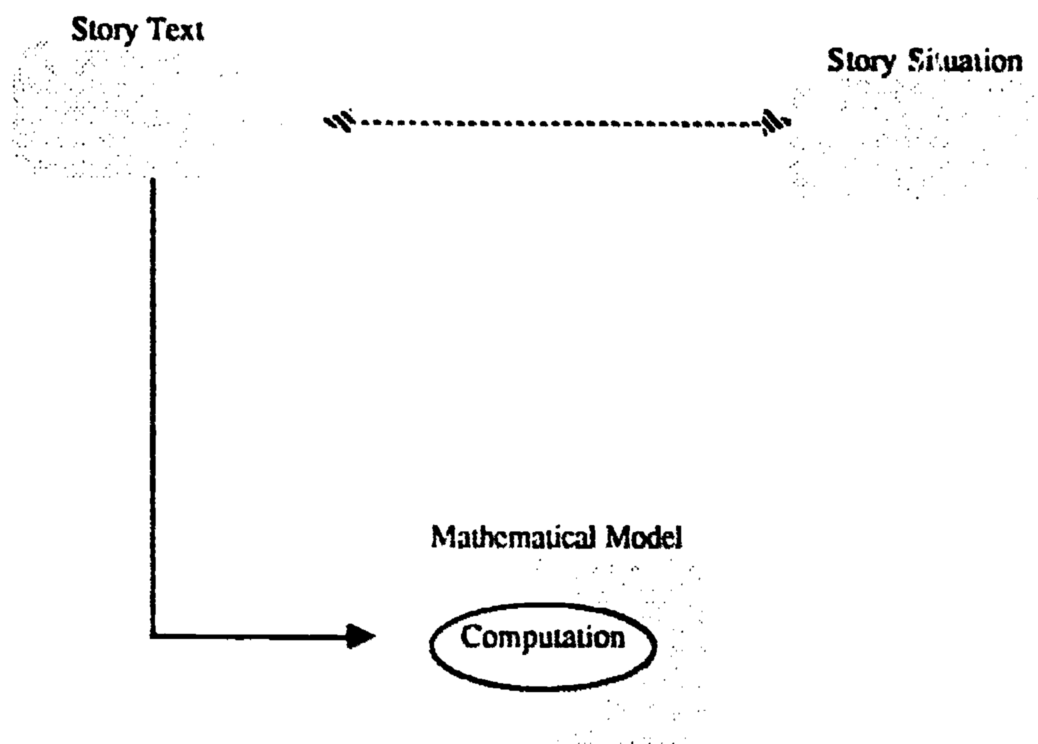

Figure 2

Schematic representation of an idealized successful solution

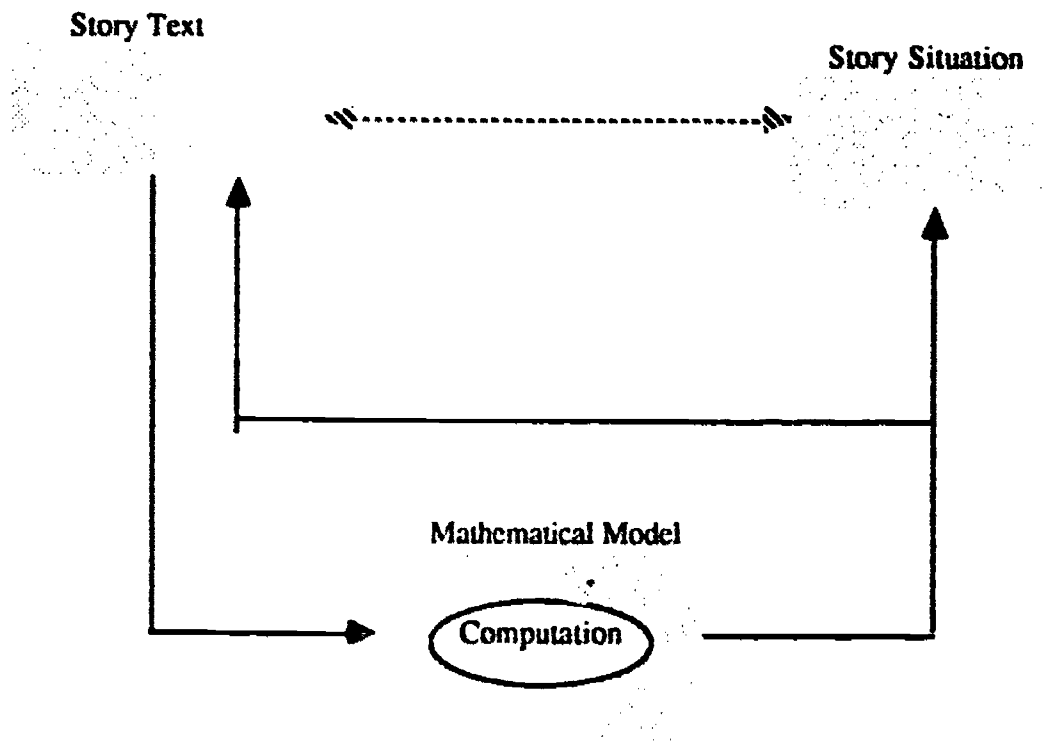

Table 1

Distribution of Solution Procedures

Procedure	Percent of Student Usage	
Appropriate Procedures		
Long Division	73%	
Repeated Multiples	7%	
Repeated Addition	5%	
Repeated Subtraction	1%	
Multiple Correct Procedures	5%	
Total		91%
Inappropriate Procedures		
Addition (Dividend + Divisor)	1%	
Subtraction (Dividend - Divisor)	1%	
Multiplication (Dividend x Divisor)	2%	
Multiple Incorrect Procedures	3%	
Total		7%
No Work Provided		2%

Table 2

Percentage of Students with Correct Procedural Execution by Procedure

	n	% with Correct Execution
Appropriate Procedures		
Long Division	142	58%
Repeated Multiples	13	77%
Repeated Addition	10	80%
Repeated Subtraction	3	67%
Multiple Correct	9	67%

Tab'e 3

Distribution of Students' Numerical Answers

Numerical Answer	Number of Students	% of students
14	84	43%
13	15	8%
13 and fractional remainder	11	6%
13 and whole number remainder	5	2%
13 and decimal remainder	2	1%
13 and other remainder representation	2	1%
Whole number remainder only	5	2%
Other Answers	62	32%
No Numerical Answer	9	5%
Total	195	100%